Mid Semestral Exam
Linear Algebra - II
Feb-2025

Time: 2 hours 30 mins

Max score: 30

Answer all questions.
Notation: V shall denote a finite dimensional vector space.

(1) (a) Prove that a square matrix N is nilpotent, namely $N^k = 0$ for some positive integer k, if and only if 0 is the only eigenvalue of N.
(b) Show that if N is a nilpotent matrix, determinant of $I + N$ is 1. (2+3)

(2) Let $T : V \to V$ be a linear operator. Define algebraic and geometric multiplicities of an eigenvalue of T. Show that for any eigenvalue of T, the geometric multiplicity is at most its algebraic multiplicity. (4)

(3) Let A and B be $n \times n$ matrices, such that $AB = BA$. Also, B has n distinct eigenvalues. Show that A and B have the same set of n linearly independent eigenvectors. (5)

(4) Show that for square matrices A and B of the same size, AB and BA have the same characteristic polynomial. Do they have the same minimal polynomial? (6)

(5) For a linear operator $T : V \to V$, show that the following are equivalent:
(a) T is diagonalizable.
(b) V admits of a basis consisting of eigenvectors of T.
(c) The minimal polynomial of T has no repeated roots.
(d) $\text{rank}(\lambda I - T) = \text{rank}(\lambda I - T)^2$ for every eigenvalue λ of T. (10)

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